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Distality Rank

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Understanding Unstable NIP Theories

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- It was motivated as an attempt to better understand unstable NIP theories by studying their stable and "purely unstable," or ***distal***, parts separately. This decomposition is particularly easy to see for algebraically closed valued fields:

Stable Part:	Residue field
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- Distality was introduced as a concept in first-order model theory by Pierre Simon in 2013.

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Stable Part:	Residue field
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- This approach can be applied to types over NIP theories where each type can be decomposed into a generically stable partial type and an order-like quotient. (Simon 2016)

Distal NIP Theories

Distality quickly became interesting and useful in its own right, and much progress has been made in recent years studying distal NIP theories. Such a theory exhibits **no stable behavior** since it is dominated by its order-like component.

Examples:

- o-minimal theories
- p-adics
- certain expansions of o-minimal theories (Hieronymi, Nell 2017)
- the asymptotic couple of the field of logarithmic transseries (Gehret, Kaplan 2018)
- the differential field of logarithmic-exponential transseries (Aschenbrenner, Chernikov, Gehret, Ziegler 2020)

Combinatorial Results

Many classical combinatorial results can be improved when study is restricted to objects definable in distal NIP structures.

- Cutting Lemma (Chernikov, Galvan, Starchenko 2018)

  > "We believe that distal structures provide the most general natural setting for investigating questions in 'generalized incidence combinatorics.'"

- (p, q)-Theorem (Boxall, Kestner 2018)

- Szemerédi Regularity Lemma (Chernikov, Starchenko 2018)

Regularity Lemma for Distal Structures (Chernikov, Starchenko 2018)

Although their result applies to infinite, as well as finite, k-partite k-uniform hypergraphs, for easier comparison to the standard Szemerédi Regularity Lemma, we state their findings for finite graphs:

Given $\mathcal{M}$ a distal NIP structure and $E \subseteq M^2$ a definable edge (i.e., symmetric and irreflexive) relation, there is a constant c such that for all finite induced graphs (V, E) and all $\varepsilon > 0$, there is a uniformly definable partition P of V with size $O(\varepsilon^{-c})$ whose defect $D \subseteq P^2$ is bounded by

$$\sum_{(A,B)\in D} |A||B| \leq \varepsilon |V|^2$$

such that the induced bipartite graph (A, B, E) on every non-defective pair $(A, B) \in P^2 \setminus D$ is homogenous (i.e., complete or empty).

Distal and non-distal NIP theories (Simon 2013)

An NIP theory is ***distal*** if and only if it has the following property: if

$$\mathcal{I}_0 + \mathcal{I}_1 + \mathcal{I}_2 \subseteq U$$

is a **dense** indiscernible sequence, where both cuts are Dedekind, and $a_0, a_1 \in U$ are such that each sequence

$$\mathcal{I}_0 + a_0 + \mathcal{I}_1 + \mathcal{I}_2$$
$$\mathcal{I}_0 + \mathcal{I}_1 + a_1 + \mathcal{I}_2$$

is indiscernible, then the sequence

$$\mathcal{I}_0 + a_0 + \mathcal{I}_1 + a_1 + \mathcal{I}_2$$

is also indiscernible.

Distal and non-distal NIP theories (Simon 2013)

Simon worked strictly **in the context of NIP theories** and proved several structural results concerning distality:

- Distality is invariant under base change; i.e.,

$$T_B \text{ is distal} \iff T \text{ is distal.}$$

- Distality can be characterized by the orthogonality of commuting global invariant types; i.e.,

  if $p(x)$ and $q(y)$ are global invariant types that commute,
  then $p(x) \cup q(y) \vdash p \otimes q$.

- It's sufficient to check one-dimensional sequences $\mathcal{I} \subset U^1$.

Distal theories can be characterized by the following property: if

$$\mathcal{I}_0 + \mathcal{I}_1 + \mathcal{I}_2 + \cdots + \mathcal{I}_{n-1} + \mathcal{I}_n$$

is an indiscernible sequence, where each cut is Dedekind, and $A = (a_0, \ldots, a_{n-1})$ is such that each sequence

$$\begin{gathered} \mathcal{I}_0 + a_0 + \mathcal{I}_1 + \mathcal{I}_2 + \cdots + \mathcal{I}_{n-1} + \mathcal{I}_n, \\ \mathcal{I}_0 + \mathcal{I}_1 + a_1 + \mathcal{I}_2 + \cdots + \mathcal{I}_{n-1} + \mathcal{I}_n, \\ \vdots \\ \mathcal{I}_0 + \mathcal{I}_1 + \mathcal{I}_2 + \cdots + \mathcal{I}_{n-1} + a_{n-1} + \mathcal{I}_n \end{gathered}$$

is indiscernible, then the sequence

$$\mathcal{I}_0 + a_0 + \mathcal{I}_1 + a_1 + \mathcal{I}_2 + a_2 + \cdots + \mathcal{I}_{n-1} + a_{n-1} + \mathcal{I}_n$$

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My research was motivated by the following questions:

Question 1 Are there theories where it is not always sufficient to check the singletons of A, but it is always sufficient to check the pairs of A?

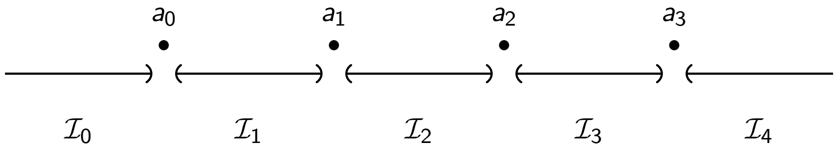
Question 2 Are there theories where it is not always sufficient to check the elements of $[A]^{m-1}$, but it is always sufficient to check the elements of $[A]^m$?

Question 3 In the existing literature, distality has been studied solely in the context of NIP theories. Is it interesting to study generalizations of distality outside of NIP?

m-Distality

1-Distality in Pictures...

A Dedekind partition $\mathcal{I} = \mathcal{I}_0 + \mathcal{I}_1 + \cdots + \mathcal{I}_4$ is ***1-distal*** iff: for all $A = (a_0, a_1, a_2, a_3)$, if each **singleton** from A inserts indiscernibly...



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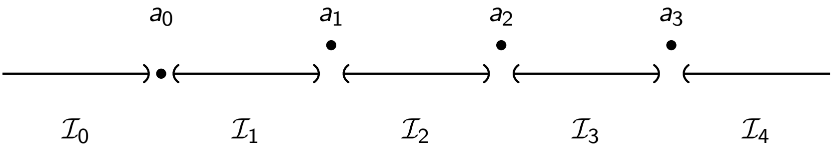

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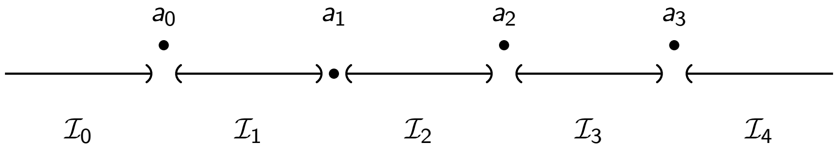

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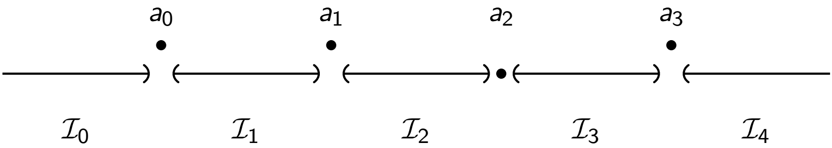

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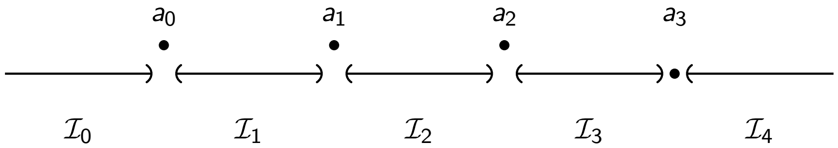

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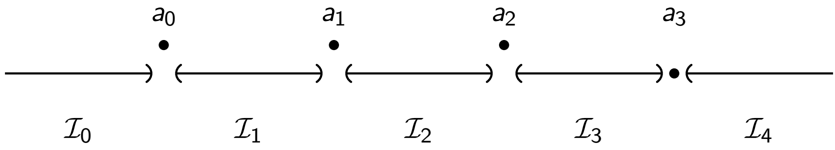

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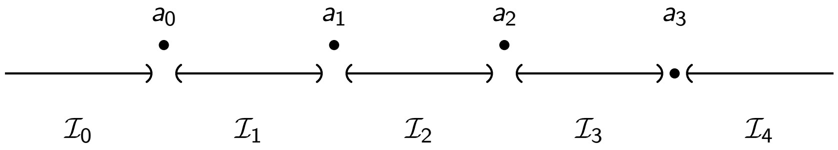

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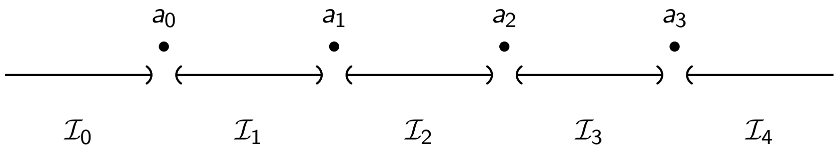

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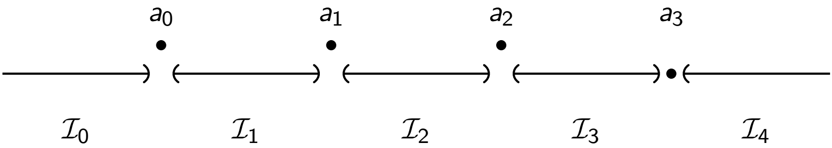

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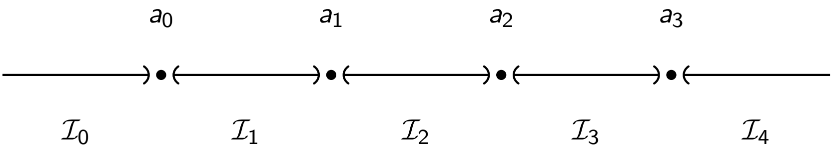

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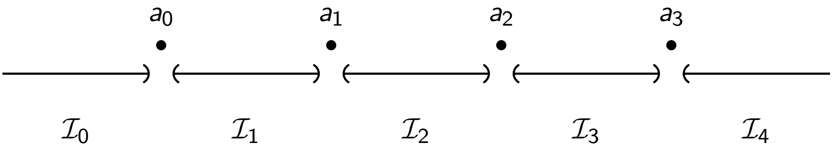

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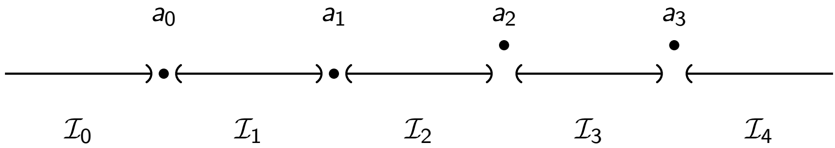

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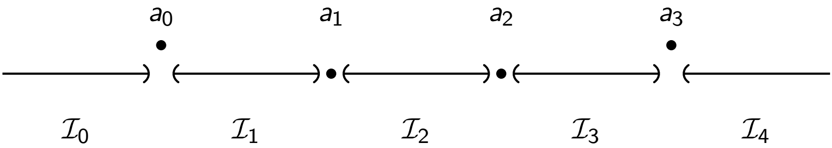

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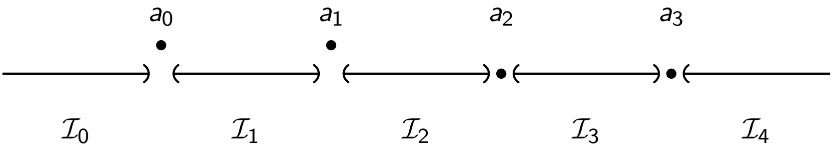

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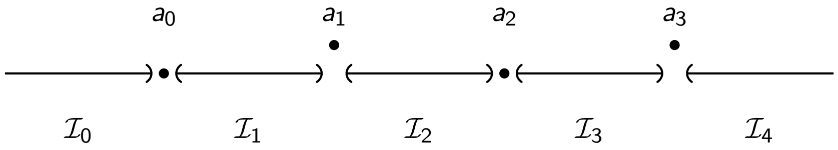

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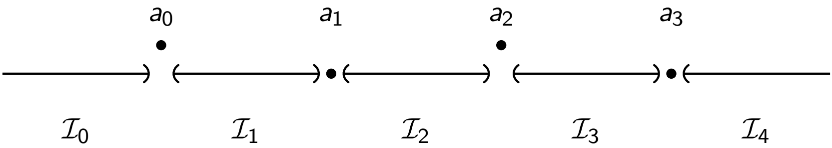

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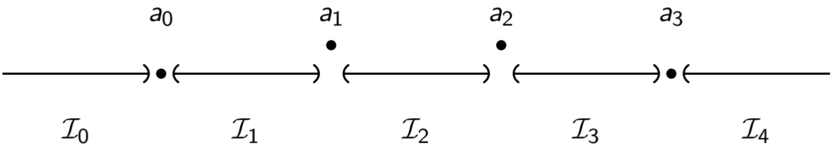

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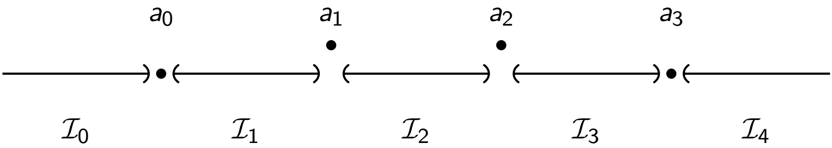

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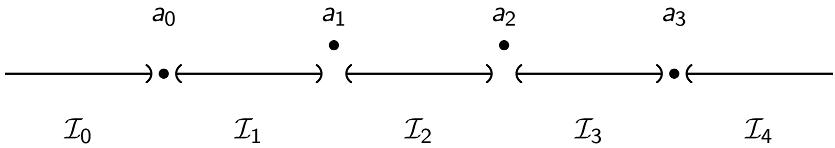

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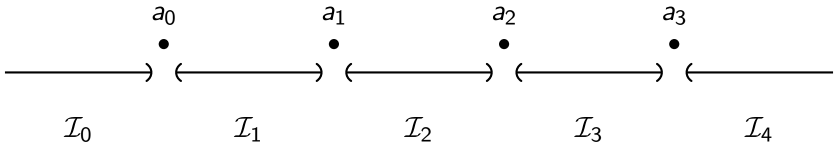

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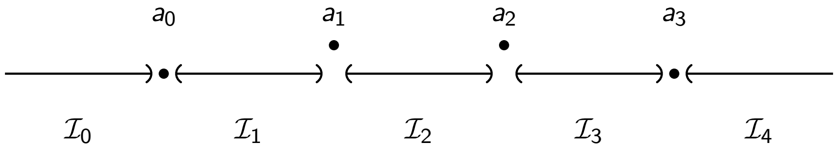

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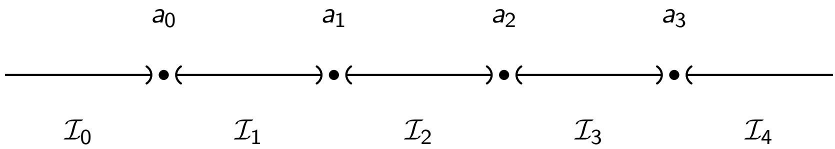

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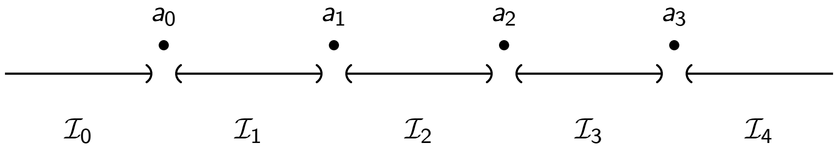

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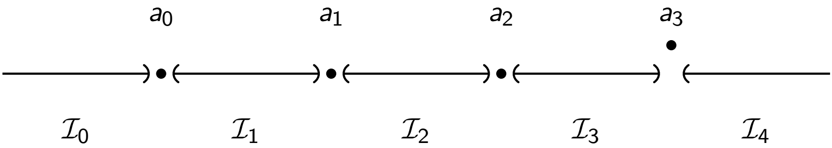

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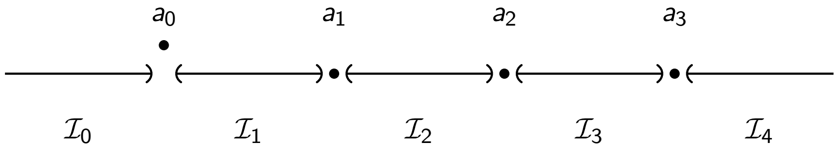

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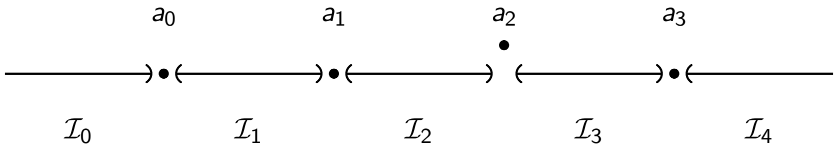

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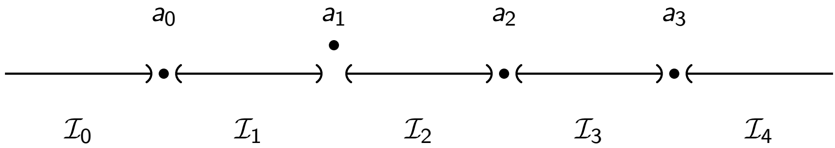

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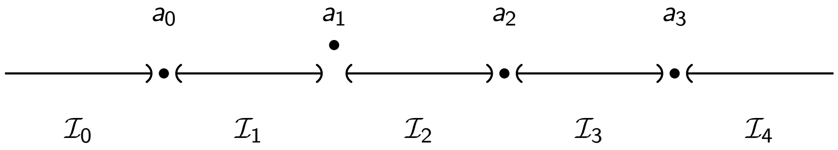

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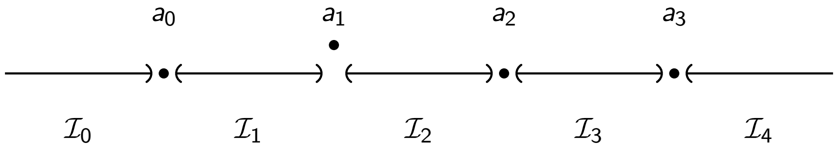

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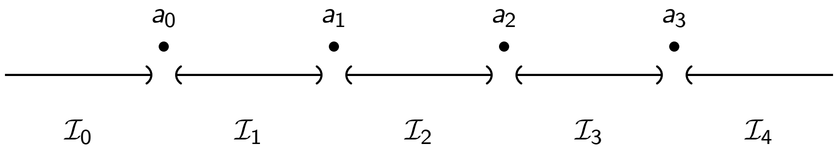

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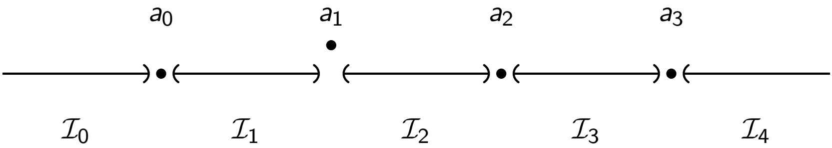

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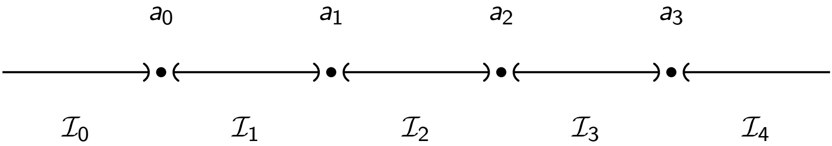

m-Distality

Let $n > m > 0$.

Definition

We say a Dedekind partition $\mathcal{I} = \mathcal{I}_0 + \cdots + \mathcal{I}_n$ is ***m-distal*** iff: for all sets $A = (a_0, \ldots, a_{n-1}) \subseteq U$, if A does not insert indiscernibly into $\mathcal{I}$, then some m-element subset of A does not insert indiscernibly into $\mathcal{I}$.

m-Distality for EM-Types

Let $n > m > 0$.

Definition

A complete EM-type Γ is ***(n, m)-distal*** iff: every Dedekind partition $\mathcal{I}_0 + \cdots + \mathcal{I}_n \models^{\text{EM}} \Gamma$ is m-distal.

Lemma

If Γ is $(m+1, m)$-distal, then Γ is (n, m)-distal for all $n > m$.

Proof: Induction on n. ∎

Definition

A complete EM-type Γ is ***m-distal*** iff: it is $(m+1, m)$-distal.

Distality Rank for EM-Types

Observation: If a complete EM-type Γ is m-distal, then it is also n-distal for all $n > m$.

Definition

The ***distality rank*** of a complete EM-type Γ, written $\mathrm{DR}(\Gamma)$, is the least $m \geq 1$ such that Γ is m-distal. If no such finite m exists, we say the distality rank of Γ is ω.

Skeletons

Let $n > m > 0$. Let $I = I_0 + \cdots + I_n$ where

$$I_0 = \omega, \quad I_1 = \omega^* + \omega, \quad \ldots \quad I_{n-1} = \omega^* + \omega, \quad I_n = \omega^*,$$

and ω^* is ω in reverse order.

Definition

If $\mathcal{I} \subseteq U$ is a sequence indexed by $I = I_0 + \cdots + I_n$, we call the corresponding partition $\mathcal{I} = \mathcal{I}_0 + \cdots + \mathcal{I}_n$ an ***n*-skeleton**.

Notice that an *n*-skeleton is a Dedekind partition with *n* cuts.

Proposition

A complete EM-*type* Γ *is m-distal if and only if there is an n-skeleton* $\mathcal{I}_0 + \cdots + \mathcal{I}_n \models^{\mathrm{EM}} \Gamma$ *which is m-distal.*

Proof

Distality Rank for Theories

Let $m > 0$.

Definition

A theory T, not necessarily complete, is ***m-distal*** iff: for all completions of T and all tuple sizes κ, every $\Gamma \in S^{\mathrm{EM}}(\kappa \cdot \omega)$ is m-distal.

In the existing literature, a theory is called distal if and only if it is 1-distal.

Definition

The ***distality rank*** of a theory T, written $\mathrm{DR}(T)$, is the least $m \geq 1$ such that T is m-distal. If no such finite m exists, we say the distality rank of T is ω.

Proposition

If T is an $\mathcal{L}$-theory with quantifier elimination and $\mathcal{L}$ contains no atomic formula with more than m free variables, then $\mathrm{DR}(T) \leq m$.

Proof: Let $\mathcal{I} = \mathcal{I}_0 + \cdots + \mathcal{I}_{m+1}$ be Dedekind and $A = (a_0, \ldots, a_m)$.

Suppose all proper subsets of A insert indiscernibly into $\mathcal{I}$.

Given $\phi \in \mathcal{L}(x_0, \ldots, x_{n-1})$, there is a T-equivalent formula

$$\bigvee_i \bigwedge_j \theta_{ij}\left(x_{\sigma_{ij}(0)}, \ldots, x_{\sigma_{ij}(m-1)}\right)$$

where each θ_{ij} is basic and each $\sigma_{ij} : m \to n$ is a function.

Let $(b_0, \ldots, b_{n-1}) \subseteq \mathcal{I}$ and $(d_0, \ldots, d_{n-1}) \subseteq \mathcal{I} \cup A$ both be increasing.

Since all m-sized subsets of A insert indiscernibly into $\mathcal{I}$, then

$$\mathcal{U} \models \theta_{ij}(b_{\sigma_{ij}(0)}, \ldots, b_{\sigma_{ij}(m-1)}) \leftrightarrow \theta_{ij}(d_{\sigma_{ij}(0)}, \ldots, d_{\sigma_{ij}(m-1)}). \quad \blacksquare$$

Finding Examples...

Corollary

Suppose $\mathcal{L}$ is a language where all function symbols are unary and all relation symbols have arity at most $m \geq 2$. If T is an $\mathcal{L}$-theory with quantifier elimination, then $\mathrm{DR}(T) \leq m$.

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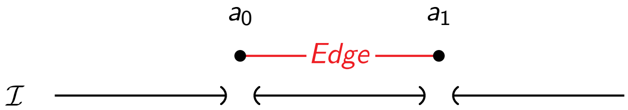
This corollary helps us find examples by putting an upper bound on distality rank:

- The theory of the random graph has distality rank 2.



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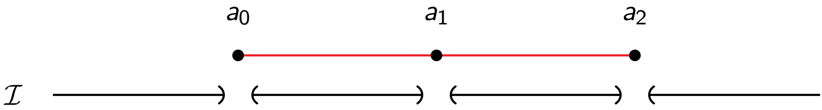

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  This generalizes, so...
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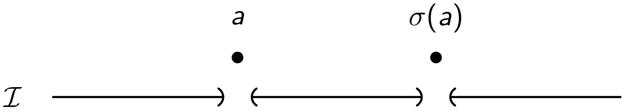
- The theory of the random m-(hyper)graph has distality rank m.
- The theories of $(\mathbb{N},\ \sigma,\ 0)$ and $(\mathbb{Z},\ \sigma)$, where $\sigma : x \mapsto x + 1$, have distality rank 2.



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We can not apply the corollary to groups...

For example, if T is the complete theory of a **strongly minimal group**, then $\mathrm{DR}(T) = \omega$:

Reset

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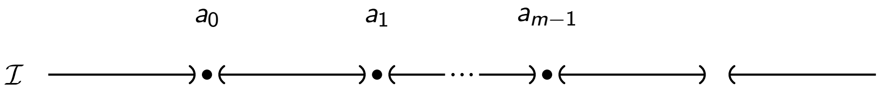

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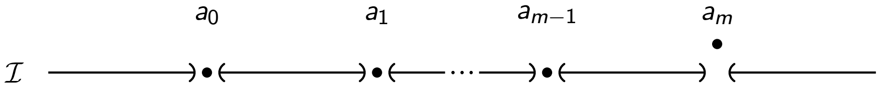

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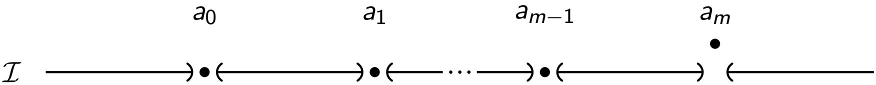
Now we can insert any m elements of A without breaking indiscernibility...



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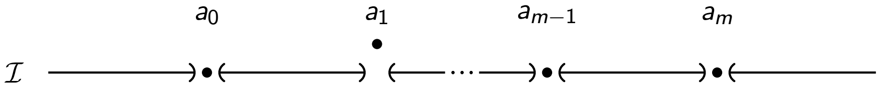

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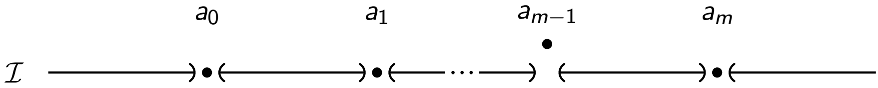

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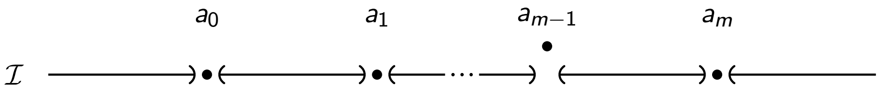

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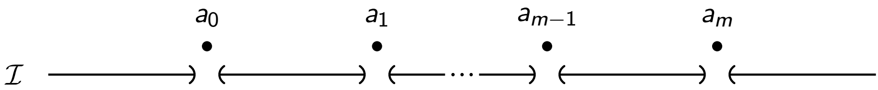

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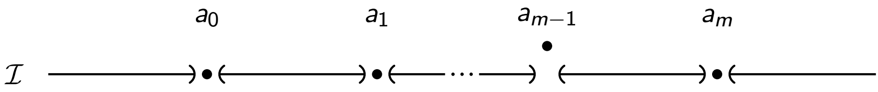

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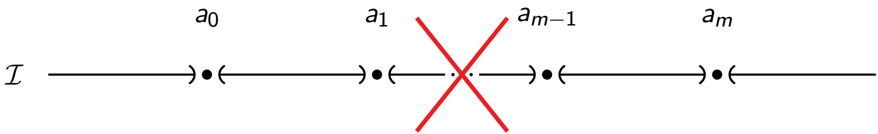

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Relationship between m-Distality and m-Dependence

Shelah introduced ***m-dependence*** as a property of first-order theories (and formulae) which generalizes NIP:

- 1-dependence $\iff$ NIP
- m-dependence $\implies$ $(m+1)$-dependence

New result courtesy of Artem Chernikov:

- m-distality $\implies$ m-dependence

Conjecture:

- m-distal regularity improves m-dependent regularity

Distal and non-distal NIP theories (Simon 2013)

Simon worked strictly **in the context of NIP theories** and proved several structural results concerning distality:

- Distality is invariant under base change; i.e.,

$$T_B \text{ is distal} \iff T \text{ is distal.}$$

- Distality can be characterized by the orthogonality of commuting global invariant types; i.e.,

  if $p(x)$ and $q(y)$ are global invariant types that commute, then $p(x) \cup q(y) \vdash p \otimes q$.

- It's sufficient to check one-dimensional sequences $\mathcal{I} \subset U^1$.

Base Change

Adding named parameters does not increase distality rank...

Proposition

If T is a complete theory and $B \subseteq \mathbb{U}$ is a small set of parameters, then $\mathrm{DR}(T_B) \leq \mathrm{DR}(T)$.

Proof:

Let $\mathcal{I} = (b_i : i \in I)$ be indiscernible over B.

Given $m > 0$, suppose there is a Dedekind partion $\mathcal{I}_0 + \ldots + \mathcal{I}_{m+1}$ of $\mathcal{I}$ and a set $A = (a_0, \ldots, a_m)$ witnessing that T_B is not m-distal.

It follows that $\mathcal{I}' = (b_i + B : i \in I)$ and $A' = (a_0 + B, \ldots, a_m + B)$ witness that T is not m-distal. ■

Base Change

If T is NIP, adding named parameters does not change distality rank...

Base Change Theorem (W. 2019)

If T is NIP and $B \subseteq U$ is a small set of parameters, then $\mathrm{DR}(T_B) = \mathrm{DR}(T)$.

Proof of Theorem:

- $\mathrm{DR}(T_B) \leq \mathrm{DR}(T)$ by the previous proposition.
- We need to show that

$$T_B \text{ is } m\text{-distal} \Rightarrow T \text{ is } m\text{-distal.}$$

But first, we need more background...

Alternation Rank

Let $\phi \in \mathcal{L}_U(x)$ and $\mathcal{I} = (b_i : i \in I) \subseteq U^{|x|}$ be an infinite indiscernible sequence.

Definition

We use $\mathrm{alt}(\phi, \mathcal{I})$ to denote the ***number of alternations of ϕ on $\mathcal{I}$***, i.e.,

$$\sup\left\{ n < \omega \ : \ \exists\, i_0 < \cdots < i_n \in I \quad \mathcal{U} \models \bigwedge_{j<n} \neg[\phi(b_{i_j}) \leftrightarrow \phi(b_{i_{j+1}})] \right\}.$$

Definition

We use $\mathrm{alt}(\phi)$ to denote the ***alternation rank of ϕ***, i.e.,

$$\sup\left\{ \mathrm{alt}(\phi, \mathcal{J}) \ : \ \mathcal{J} \subseteq U^{|x|} \text{ is an infinite indiscernible sequence} \right\}.$$

IP and NIP

Definition

A formula $\phi \in \mathcal{L}(x, y)$ is ***IP*** iff: there is a $d \in U^{|y|}$ such that $\text{alt}(\phi(x, d)) = \infty$.

Definition

The theory T is ***IP*** iff: there is a $\phi \in \mathcal{L}_U(x)$ with $\text{alt}(\phi) = \infty$.

In both cases, we use ***NIP*** to denote the, often more desirable, condition of not being IP.

Limit Types

Let $(I, <)$ be a linear order and let $\mathcal{I} = (b_i : i \in I) \subseteq U$ be a sequence of tuples.

Definition

Given $A \subseteq U$, if the partial type

$$\{\phi \in \mathcal{L}_A(x) : \exists i \in I \, \forall j \geq i \;\; \mathcal{U} \models \phi(b_j)\}$$

is complete, we call it the ***limit type of $\mathcal{I}$ over A***, written $\operatorname{limtp}_A(\mathcal{I})$. Moreover, if it exists, we call $\operatorname{limtp}_U(\mathcal{I})$ the ***global limit type of $\mathcal{I}$*** and may simply write $\lim(\mathcal{I})$.

- If $\mathcal{I}$ is indiscernible, then $\operatorname{limtp}_{\mathcal{I}}(\mathcal{I})$ exists.
- If T is NIP and $\mathcal{I}$ is indiscernible, the global limit type $\lim(\mathcal{I})$ exists.

In order to prove the Base Change Theorem, we need the following lemma...

Let $m > 0$.

Base Change Lemma

Suppose T is NIP. If

- $\mathcal{I} = \mathcal{I}_0 + \cdots + \mathcal{I}_{m+1}$ is a Dedekind partition,
- $A = (a_0, \ldots, a_m)$ is a set of parameters such that every proper subset inserts indiscernibly into $\mathcal{I}$, and
- $D \subseteq U$ is a small set of parameters,

then there is a set $A' = (a'_0, \ldots, a'_m)$ such that $A' \equiv_{\mathcal{I}} A$ and for each $\sigma : m \to m+1$ increasing, we have

$$a'_{\sigma(0)} \cdots a'_{\sigma(m-1)} \models \operatorname{limtp}_D \left(\mathfrak{c}^-_{\sigma(0)}, \ldots, \mathfrak{c}^-_{\sigma(m-1)} \right).$$

Proof

Now we can prove the Base Change Theorem...

Base Change Theorem (W. 2019)

If T is NIP and $B \subseteq U$ is a small set of parameters, then $\mathrm{DR}(T_B) = \mathrm{DR}(T)$.

Proof of Theorem (continued):

It remains to show that T_B is m-distal $\Rightarrow$ T is m-distal.

Suppose $\Gamma \in S^{\mathrm{EM}}$ is not m-distal.

Let $\mathcal{I}_0 + \cdots + \mathcal{I}_{m+1} \models^{\mathrm{EM}} \Gamma$ be a skeleton which is indiscernible over B.

There exists a set $A = (a_0, \ldots, a_m)$ such that every proper subset inserts indiscernibly over $\varnothing$ but A does not.

Applying the lemma with $D = B \cup \mathcal{I}$ yields a set A' such that every proper subset inserts indiscernibly over B but A' does not. ■

Theorem (W.)

Suppose T is NIP. A complete EM-type Γ is m-distal if and only if there is an m-distal Dedekind partition $\mathcal{I}_0 + \cdots + \mathcal{I}_{m+1} \models^{\mathrm{EM}} \Gamma$.

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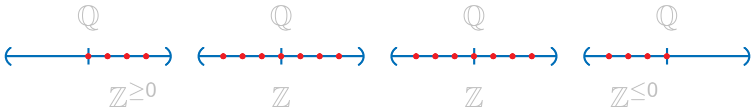

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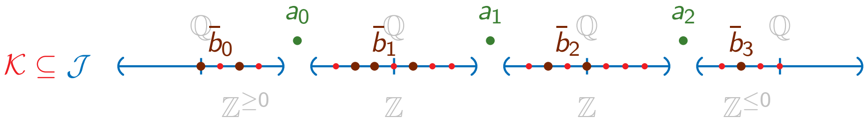

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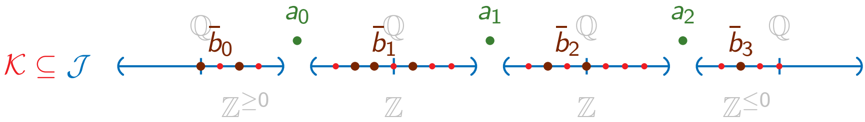

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$$a'_{\sigma(0)} \cdots a'_{\sigma(m-1)} \models \operatorname{limtp}_{\mathcal{J}} \left(\mathfrak{c}^-_{\sigma(0)}, \ldots, \mathfrak{c}^-_{\sigma(m-1)} \right).$$

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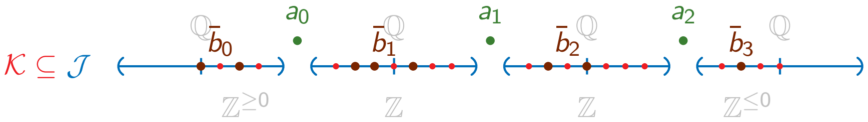

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It follows that (ϕ, A', B) witnesses that $\mathcal{J}$ is not m-distal. ■

Distal and non-distal NIP theories (Simon 2013)

Simon worked strictly **in the context of NIP theories** and proved several structural results concerning distality:

- Distality is invariant under base change; i.e.,

$$T_B \text{ is distal } \iff T \text{ is distal.}$$

- Distality can be characterized by the orthogonality of commuting global invariant types; i.e.,

  if $p(x)$ and $q(y)$ are global invariant types that commute, then $p(x) \cup q(y) \vdash p \otimes q$.

- It's sufficient to check one-dimensional sequences $\mathcal{I} \subset U^1$.

Type Determinacy

Let $n > m > 0$.

Definition

Given $p \in S_A(x_0, \ldots, x_{n-1})$, we say that the n-type p is ***m-determined*** iff: it is completely determined by the m-types

$$\{q \in S_A(x_{i_0}, \ldots, x_{i_{m-1}}) : q \subseteq p \text{ and } i_0 < \cdots < i_{m-1} < n\}$$

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Theorem (W. 2019)

If T is m-distal, then for any n global invariant types

$$p_0(x_0), \ldots, p_{n-1}(x_{n-1})$$

which commute pairwise, their product $p_0 \otimes \cdots \otimes p_{n-1}$ is m-determined.

***Furthermore**, if T is NIP, the converse holds as well.*

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Example: For higher distality ranks, it's not always enough to check sequences of singletons.

Let $\mathcal{L} = \{R, <, P_0, P_1\}$. Let T be the complete theory of an ***ordered random bipartite graph***; i.e. the theory axiomatized by the following:

1. All models are linearly ordered by $<$.
2. The ordering is partitioned by $P_0 < P_1$ where each part has no endpoints.
3. All models are bipartite graphs, with parts P_0, P_1 and edge relation R.
4. For each $s, t < \omega$ and each $i < 2$, we have the following axiom:

$\forall$ distinct $x_0, \ldots, x_{s-1}, y_0, \ldots, y_{t-1} \in P_i \quad \forall z_0 < z_1 \in P_{1-i} \quad \exists z \in P_{1-i}$

$$\left[z_0 < z < z_1 \wedge \bigwedge_{r<s} x_r \mathrel{\mathrm{R}} z \;\wedge\; \bigwedge_{r<t} y_r \not\mathrel{\mathrm{R}} z \right]$$

Example: For higher distality ranks, it's not always enough to check sequences of singletons.

Let $\mathcal{L} = \{R, <, P_0, P_1\}$. Let T be the complete theory of an ***ordered random bipartite graph***; i.e. the theory axiomatized by the following:

1. All models are linearly ordered by $<$.
2. The ordering is partitioned by $P_0 < P_1$ where each part has no endpoints.
3. All models are bipartite graphs, with parts P_0, P_1 and edge relation R.
4. For each $s, t < \omega$ and each $i < 2$, we have the following axiom:

$\forall$ distinct $x_0, \ldots, x_{s-1}, y_0, \ldots, y_{t-1} \in P_i \quad \forall z_0 < z_1 \in P_{1-i} \quad \exists z \in P_{1-i}$

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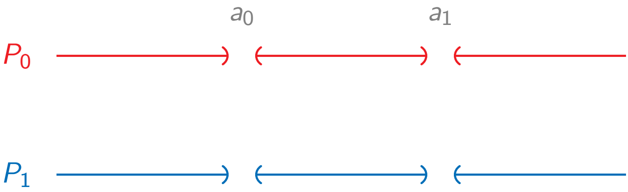

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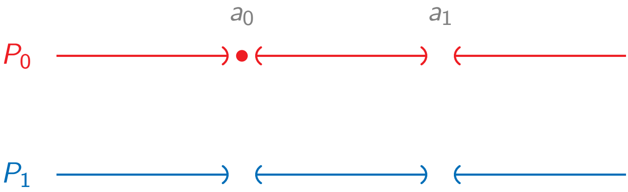

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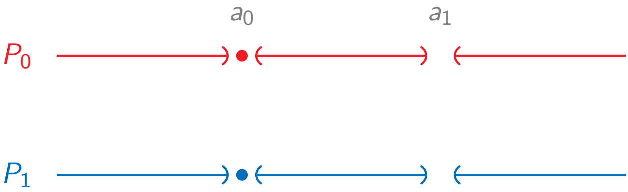

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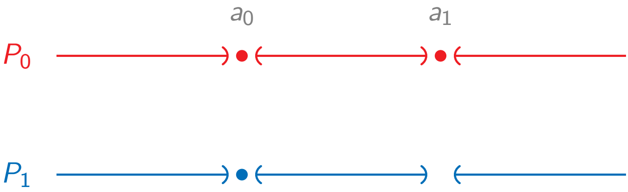

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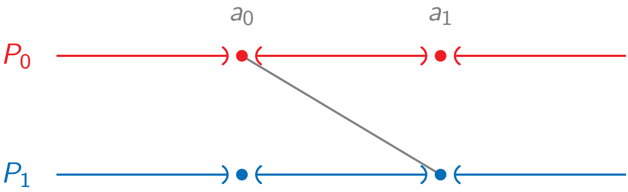
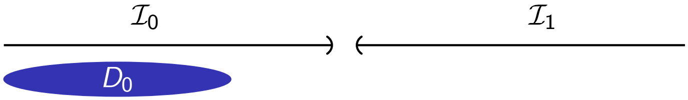


Reset

Strong m-Distality

A Dedekind partition $\mathcal{I} = \mathcal{I}_0 + \mathcal{I}_1$ is ***strongly 1-distal*** iff: for all small bases $D_0 \subseteq U$, if $\mathcal{I}$ is indiscernible over D_0



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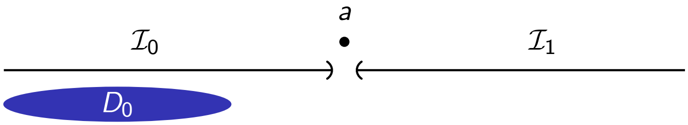

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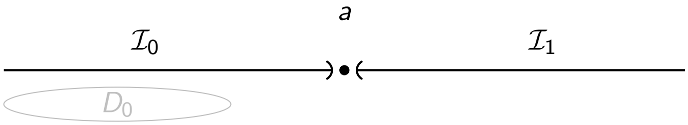

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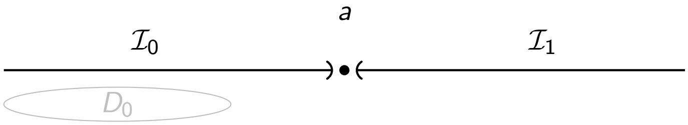

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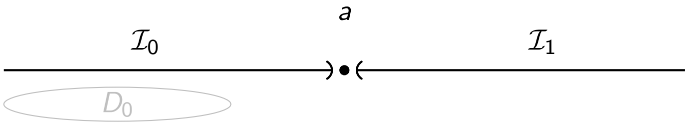

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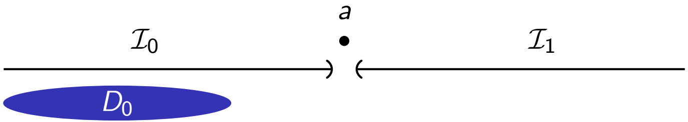

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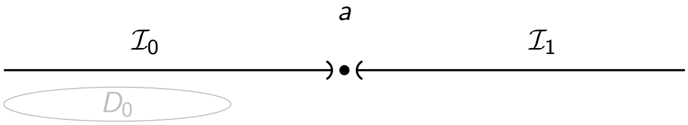

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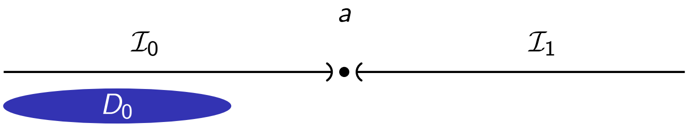

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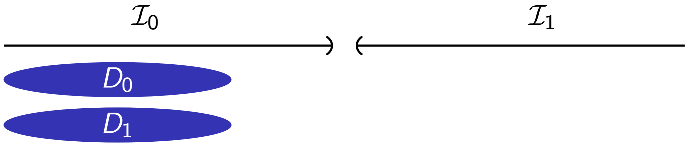

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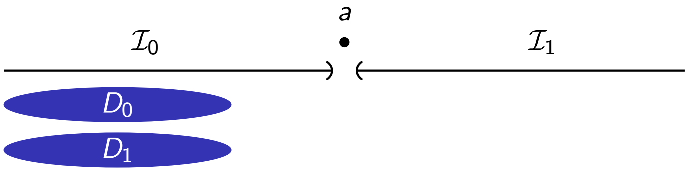

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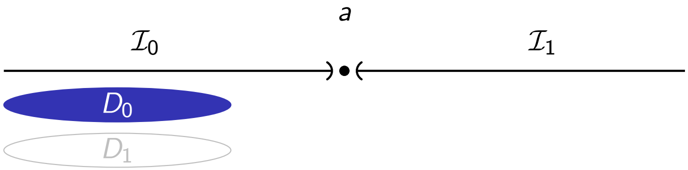

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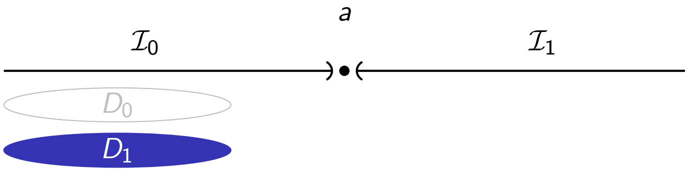

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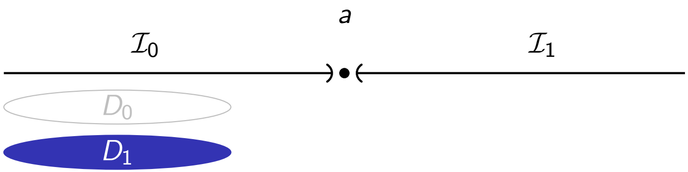

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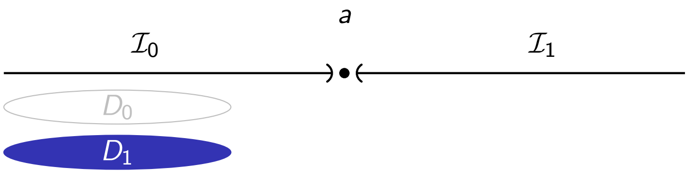

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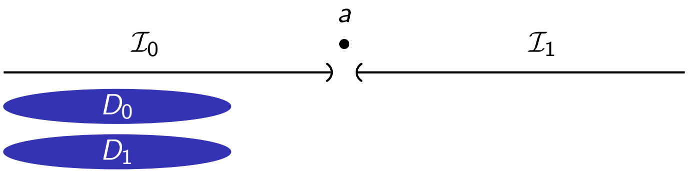

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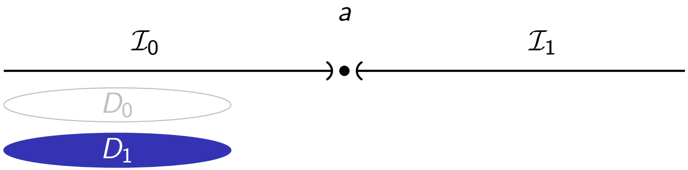

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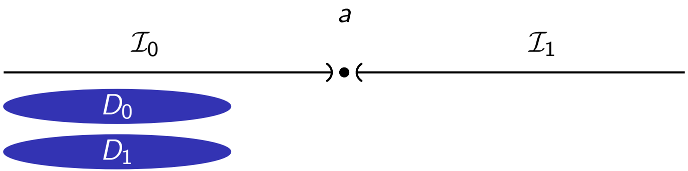

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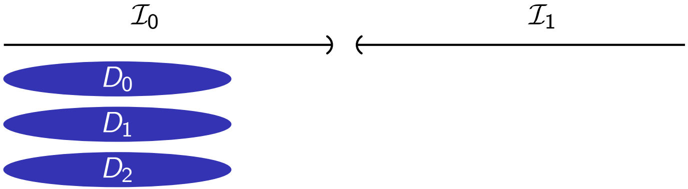

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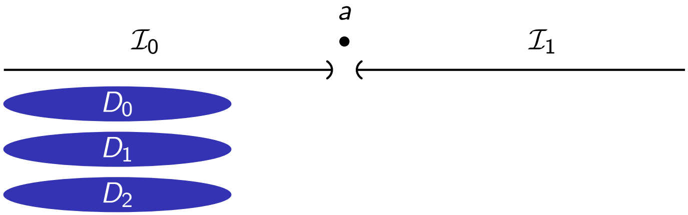

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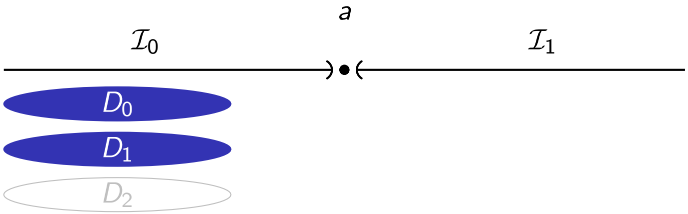

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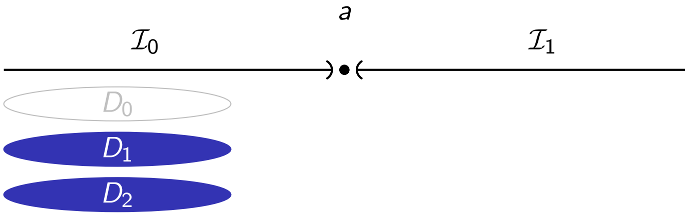

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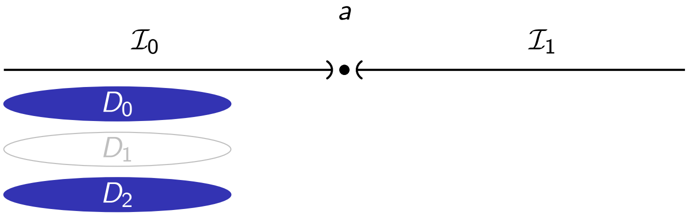

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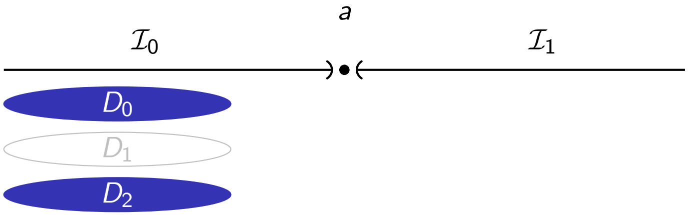

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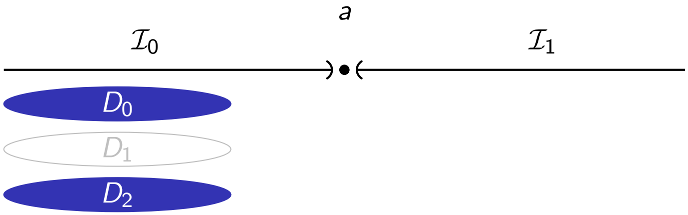

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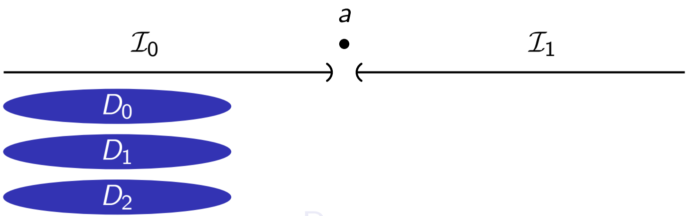

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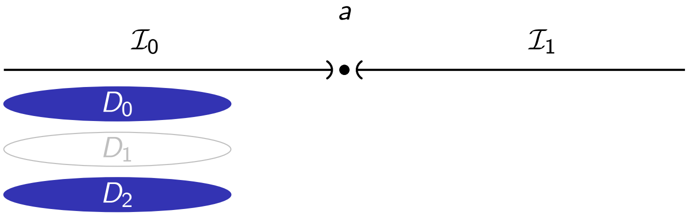

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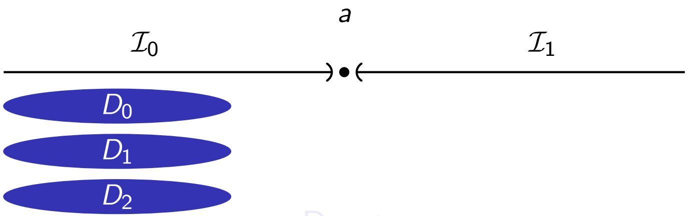


Reset

Let $m > 0$.

Definition

An indiscernible Dedekind partition $\mathcal{I}_0 + \mathcal{I}_1$ is ***strongly m-distal*** iff: for all $a \in U$ and all sequences of small sets $\overline{D} = (D_0, \ldots, D_{m-1})$, if $\mathcal{I}_0 + \mathcal{I}_1$ is indiscernible over $\overline{D}$ and $\mathcal{I}_0 + a + \mathcal{I}_1$ is indiscernible over $\bigcup_{i \neq j} D_i$ for all $j < m$, then $\mathcal{I}_0 + a + \mathcal{I}_1$ is indiscernible over $\overline{D}$.

Definition

A complete EM-type Γ is ***strongly m-distal*** iff: all Dedekind partitions $\mathcal{I}_0 + \mathcal{I}_1 \models^{\mathrm{EM}} \Gamma$ are strongly m-distal.

Definition

The ***strong distality rank*** of a complete EM-type Γ, written $\mathrm{SDR}(\Gamma)$, is the least $m \geq 1$ such that Γ is strongly m-distal. If no such finite m exists, we say the strong distality rank of Γ is ω.

Lemma

Suppose $\Gamma \in S^{\mathrm{EM}}$ *is not strongly m-distal and* $\mathcal{I} = \mathcal{I}_0 + \mathcal{I}_1 \models^{\mathrm{EM}} \Gamma$ *is a Dedekind partition indexed by* $(I_0 + I_1, <)$. *There is a witness* $(\overline{D}, \phi, a)$ *where*

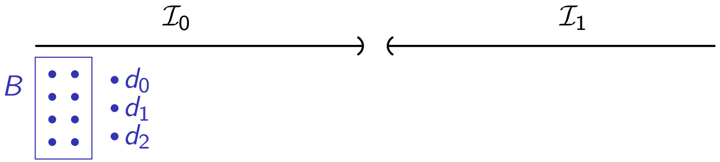
- $\overline{D} = (D_0, \ldots, D_{m-1})$ *is such that* $\mathcal{I}$ *is indiscernible over* $\overline{D}$,
- $\phi(x) \in \mathrm{tp}^{\mathrm{EM}}_{\overline{D}}(\mathcal{I})$, *and*
- $a \in U$ *is such that* $\mathcal{I}_0 + a + \mathcal{I}_1$ *is indiscernible over* $\bigcup_{i \neq j} D_i$ *for all* $j < m$ *but* $\mathcal{U} \not\models \phi(a)$.

Moreover, we may assume that $\overline{D} = (Bd_0, \ldots, Bd_{m-1})$ *for some finite base* $B \subseteq U$ *and singletons* $d_0, \ldots, d_{m-1} \in U^1$ *and that* $\mathcal{I}_0 + a + \mathcal{I}_1$ *is indiscernible over* $B \cup \{d_i : i \neq j\}$ *for each* $j < m$.

Corollary

A complete EM-*type* Γ *is strongly m-distal if and only if there is a Dedekind partition* $\mathcal{I}_0 + \mathcal{I}_1 \models^{\mathrm{EM}} \Gamma$ *which is strongly m-distal.*

Suppose a Dedekind partition $\mathcal{I} = \mathcal{I}_0 + \mathcal{I}_1$ is ***strongly 3-distal***. If $\mathcal{I}$ is indiscernible over $Bd_0d_1d_2$



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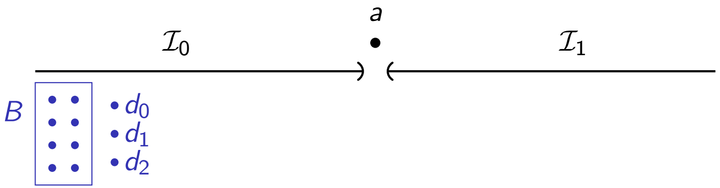

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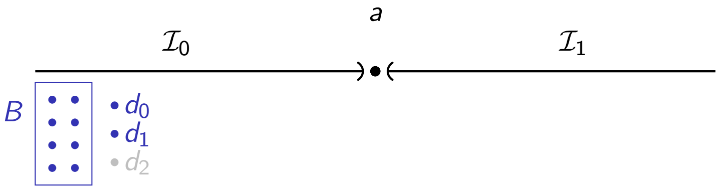

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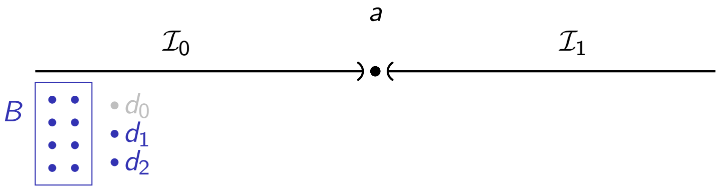

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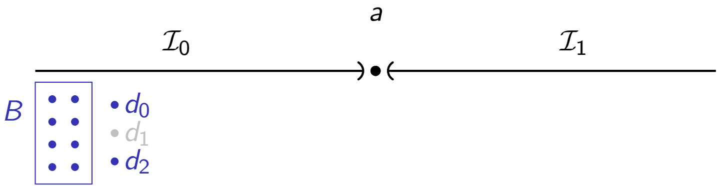
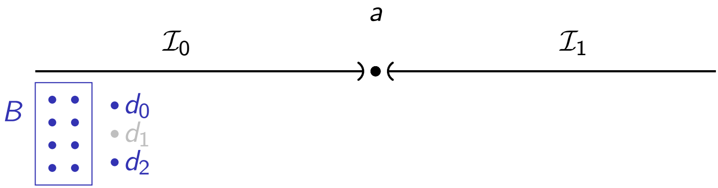


Reset

Suppose a Dedekind partition $\mathcal{I} = \mathcal{I}_0 + \mathcal{I}_1$ is ***strongly 3-distal***. If $\mathcal{I}$ is indiscernible over $Bd_0d_1d_2$ and $a \in U$ inserts over Bd_id_j for $i < j < 3$...



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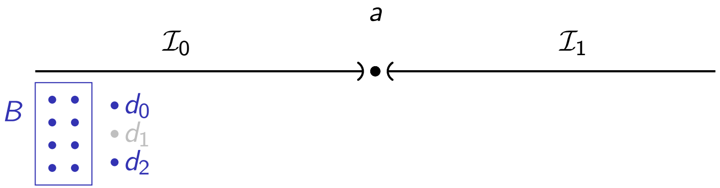

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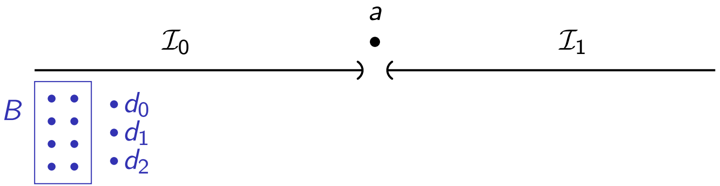

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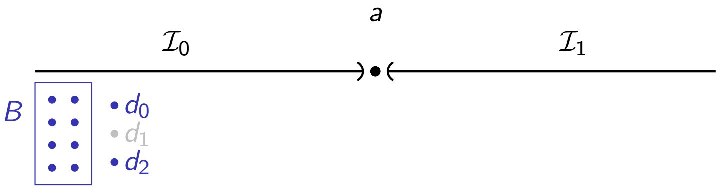

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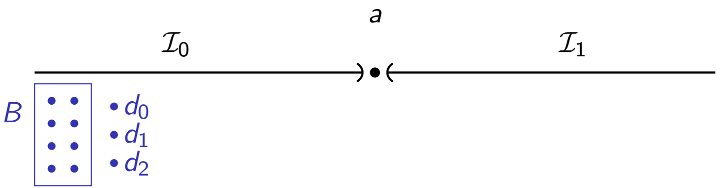

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Strong m-Distality $\implies$ m-Distality

Let $m > 0$.

Proposition

Suppose a complete EM*-type* Γ *is strongly m-distal. If a Dedekind parition* $\mathcal{I}_0 + \cdots + \mathcal{I}_{m+1} \models^{\mathrm{EM}} \Gamma$ *is indiscernible over some small set* B *and* $A = (a_0, \ldots, a_m)$ *is such that every proper subset inserts indiscernibly over* B*, then* A *inserts indiscernibly over* B*. In particular,* Γ *is m-distal.*

Proof:

Let $D_i = B\mathcal{I}_i a_i$ for each $i < m$.

Since Γ is strongly m-distal, it follows that $\mathcal{I}_m + a_m + \mathcal{I}_{m+1}$ is indiscernible over $\overline{D}$. ■

Example: $\mathrm{DR}(\Gamma) < \mathrm{SDR}(\Gamma)$

Let $\mathcal{L} = \{R, <, P_0, P_1\}$ with R binary, and let T be the theory of the ***ordered random bipartite graph***. If Γ is the EM-type of an increasing sequence of singletons in P_0, the $\mathrm{DR}(\Gamma) = 1 \ldots$

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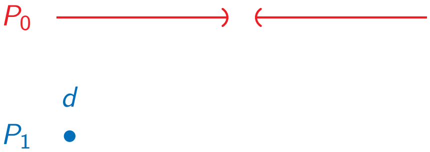

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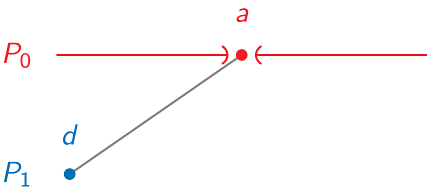


Proposition

If T is an $\mathcal{L}$-theory with quantifier elimination and $\mathcal{L}$ contains no atomic formula with more than m free variables, then $\mathrm{SDR}(T) \leq m$.

Corollary

Suppose $\mathcal{L}$ is a language where all function symbols are unary and all relation symbols have arity at most $m \geq 2$. If T is an $\mathcal{L}$-theory with quantifier elimination, then $\mathrm{SDR}(T) \leq m$.

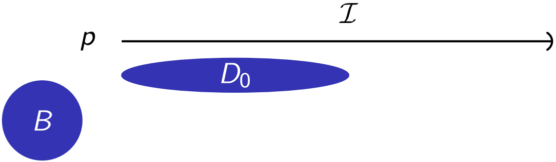
For the following examples, distality rank and strong distality rank agree:

- The theory of the random m-hypergraph has strong distality rank m.
- The theories of $(\mathbb{N}, \sigma, 0)$ and $(\mathbb{Z}, \sigma)$, where $\sigma : x \mapsto x + 1$, have strong distality rank 2.
- If T is the complete theory of a strongly minimal group, then $\mathrm{SDR}(T) = \omega$.

Strong m-Distality for Invariant Types

Let $p \in S_U(x)$ be a global type which is invariant over some small set of parameters $B \subseteq U$.

p is ***strongly 1-distal over*** B iff: for all small bases $D_0 \subseteq U$ and all Morley sequences $\mathcal{I} \models p^\omega\restriction_{BD_0}$,



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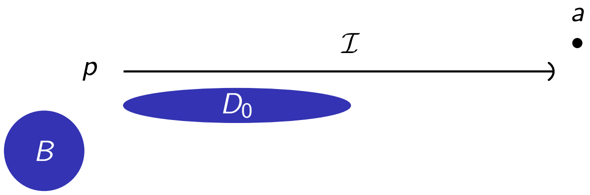
p is ***strongly 1-distal over** B* iff: for all small bases $D_0 \subseteq U$ and all Morley sequences $\mathcal{I} \models p^\omega|_{BD_0}$, if $a \in U$ extends the sequence over B...



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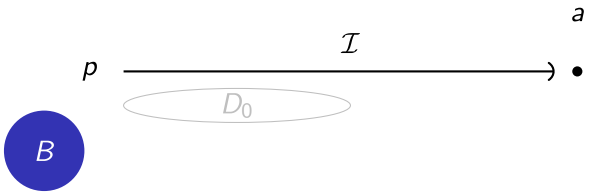

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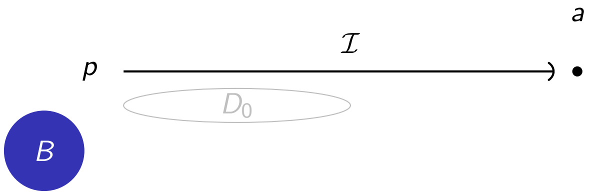

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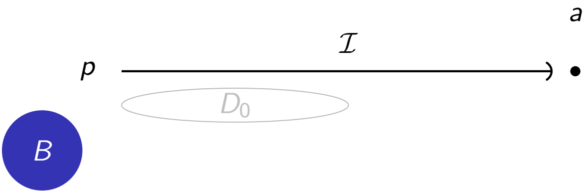

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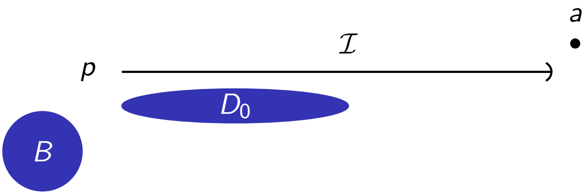

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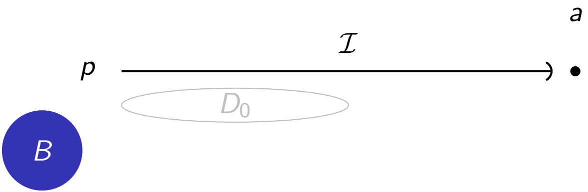

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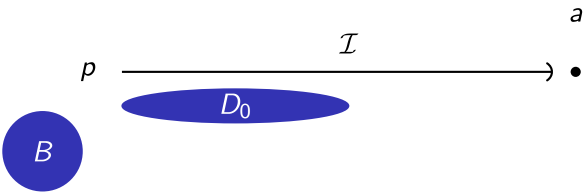

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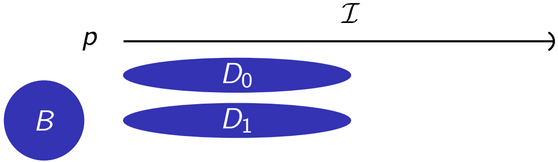

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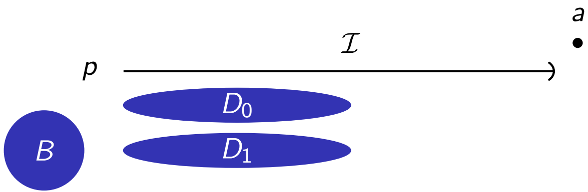

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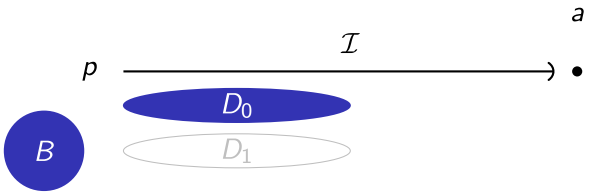

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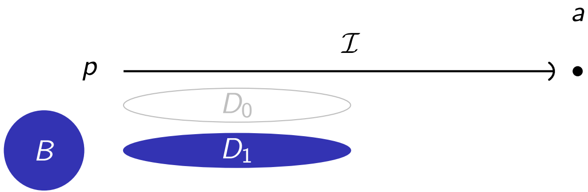

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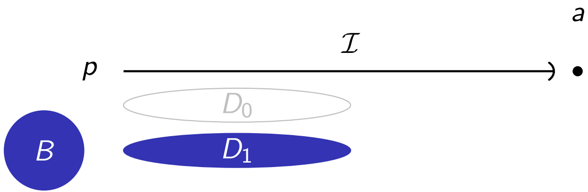

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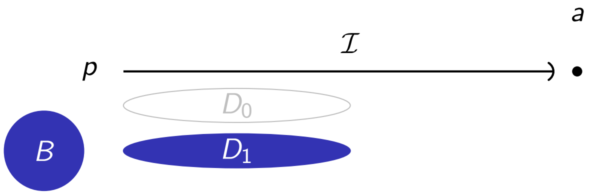

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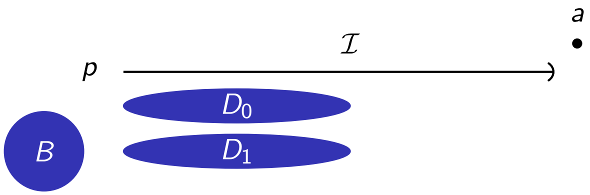

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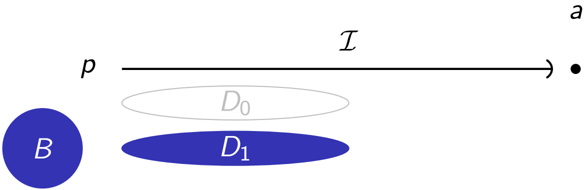

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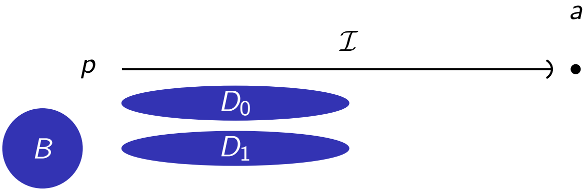

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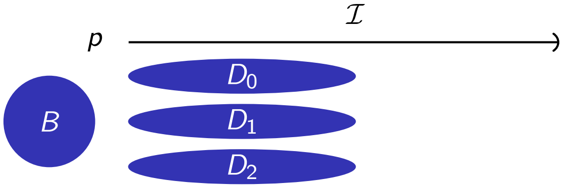

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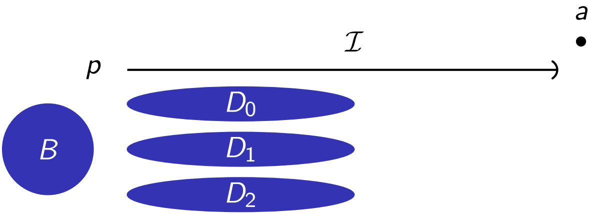

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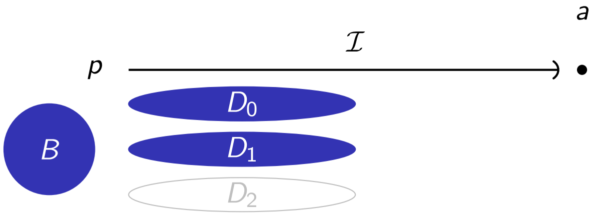

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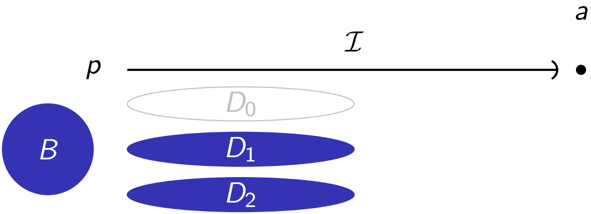

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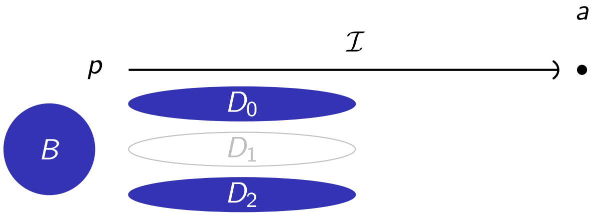

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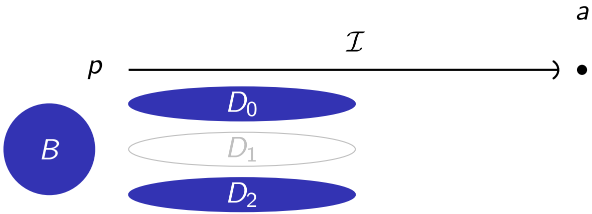

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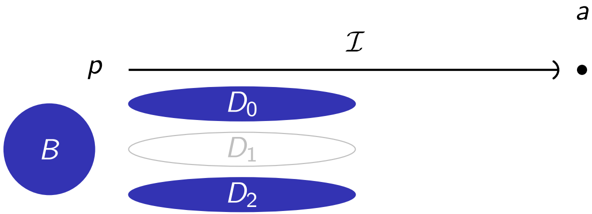

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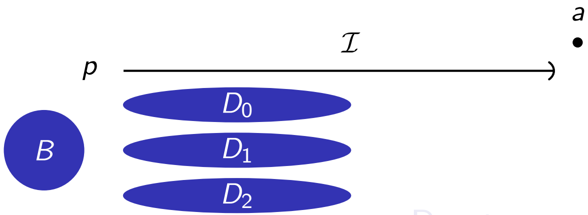

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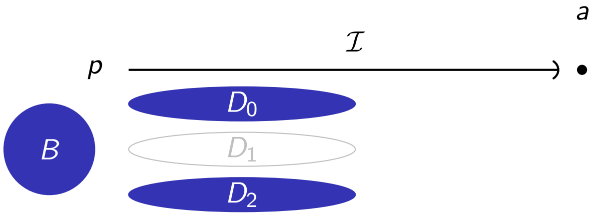

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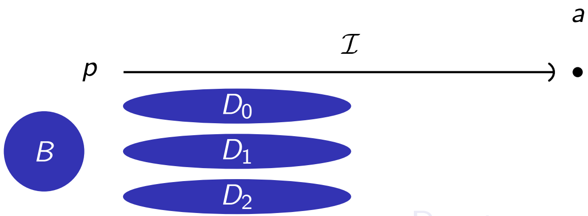

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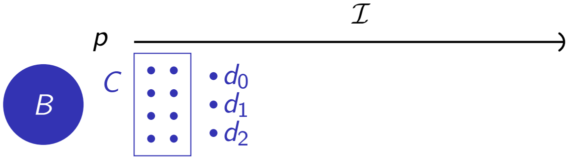
Lemma

Let $m > 0$. If p is not strongly m-distal over B and $\mathcal{I}$ is an infinite Morley sequence for p over B with no last element, then there is a witness $(\bar{D}, \phi, a)$ where

- $\bar{D} = (D_0, \ldots, D_{m-1})$ *is such that* $\mathcal{I} \models p^{\omega}\lfloor_{B\bar{D}}$,
- $\phi(x) \in p\lfloor_{B\bar{D}}$, *and*
- $a \in U^x$ *is such that* $\mathcal{I} + a$ *is indiscernible over* $B \cup \bigcup_{i \neq j} D_j$ *for each* $j < m$ *but* $\mathcal{U} \models \neg\phi(a)$.

Moreover, we may assume that $\bar{D} = (Cd_0, \ldots, Cd_{m-1})$ for some finite $C \subseteq U$ and singletons $d_0, \ldots, d_{m-1} \in U^1$ and that $\mathcal{I} + a$ is indiscernible over $BC\bar{d} \setminus d_j$ for each $j < m$.

p is ***strongly 3-distal*** iff: for all finite $C \subseteq U$, all singletons $d_0, d_1, d_2 \in U$ and all Morley sequences $\mathcal{I} \models p^\omega|_{BC\bar{d}}$,



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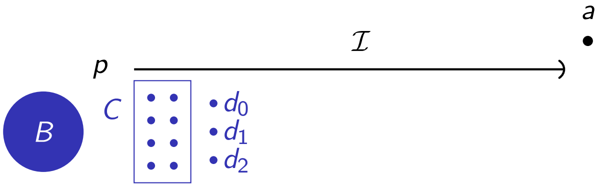

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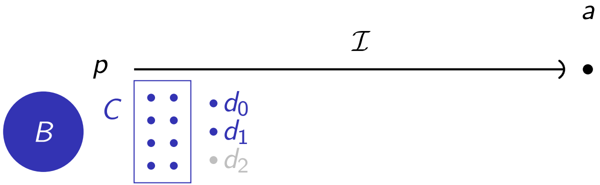

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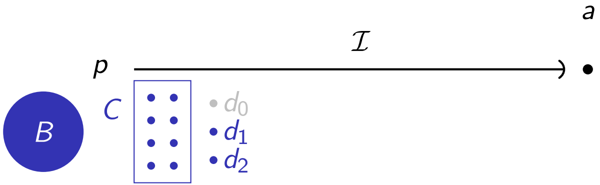

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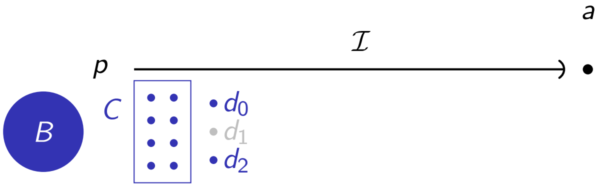

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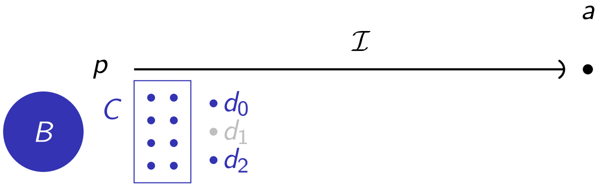

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Reset

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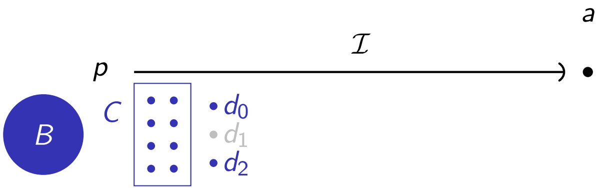

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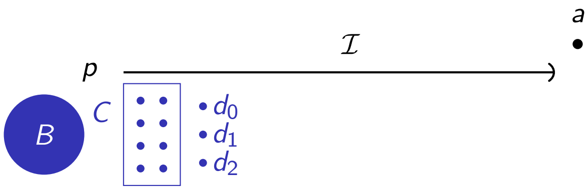

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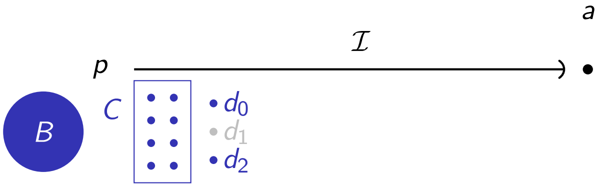

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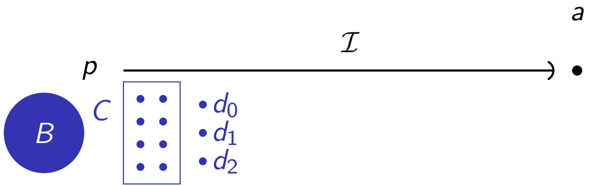

Reset

Strong Distality Rank for Invariant Types

Definition

The ***strong distality rank*** of p, written $\mathrm{SDR}(p)$, is the least $m \geq 1$ such that p is strongly m-distal. If no such finite m exists, we say the strong distality rank of p is ω.

Strong Distality and m-Determinacy

Let $n \geq m > 0$.

Proposition

Given a global invariant type $q \in S_U(y_0, \ldots, y_{n-1})$, if p is strongly m-distal over A and $p \otimes q = q \otimes p$, then the product is determined by q and restrictions of the form

$$(p \otimes q)\lfloor_{xy_{\sigma(0)} \cdots y_{\sigma(m-2)}}$$

where $\sigma : (m-1) \to n$.

Strong Distality Rank and Geometric Stability

Let T be a complete **strongly minimal theory**.

Let $g \in S_U(1)$ be the generic global type.

Let $m > 0$.

Proposition

The generic type g is strongly m-distal if and only if for every $A \subseteq U$, we have

$$\operatorname{acl}(A) \quad = \quad \bigcup_{A' \in [A]^{<m}} \operatorname{acl}(A')$$

where $[A]^{<m}$ denotes all subsets $A' \subseteq A$ with $|A'| < m$.

Strong Distality Rank and Geometric Stability

The proposition has the following geometric implications...

(I) $\mathrm{SDR}(g) = 2 \iff (U, \mathrm{acl})$ is trivial

(II) $\mathrm{SDR}(g) \leq 3 \implies (U, \mathrm{acl})$ is modular

(III) $\mathrm{SDR}(g) < \omega \implies (U, \mathrm{acl})$ is locally modular

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(IV) $\mathrm{SDR}(g) = \omega \impliedby (U, \mathrm{acl})$ is non-trivial

It follows that **(II)** and **(III)** are vacuous.

Strong Distality Rank and Geometric Stability

We are left with...

(I) $\mathrm{SDR}(g) = 2 \iff (U, \mathrm{acl})$ is trivial

(IV) $\mathrm{SDR}(g) = \omega \impliedby (U, \mathrm{acl})$ is non-trivial

Which combine to yield the following theorem...

Theorem (W.)

If (U, acl) is trivial, then

$$\mathrm{SDR}(g) = 2.$$

If not, then

$$\mathrm{SDR}(g) = \omega.$$

Thank You!

A link to the paper and other interesting things can be found at my website...

https://homepages.math.uic.edu/~roland/

Appendix

Proof:

($\Leftarrow$) Suppose Γ is not m-distal. Let $\mathcal{I} = \mathcal{I}_0 + \cdots + \mathcal{I}_{m+1} \models^{\mathrm{EM}} \Gamma$ be an $(m+1)$-skeleton. We will show that the skeleton is not m-distal.

Since Γ is not m-distal, there exist $\mathcal{J} \models^{\mathrm{EM}} \Gamma$, a Dedekind partition $\mathcal{J} = \mathcal{J}_0 + \cdots + \mathcal{J}_{m+1}$, and a sequence $A = (a_0, \dots, a_m) \in U$ such that all m-sized subsets insert but A does not. Let $\phi \in \Gamma$ and $\bar{b}_i \in \mathcal{J}_i$ such that

$$\mathcal{U} \not\models \phi(\bar{b}_0, a_0, \dots, \bar{b}_m, a_m, \bar{b}_{m+1}).$$

Construct $\sigma : \mathcal{I} \to \mathcal{J}$ an order-preserving map such that

$$\bar{b}_i \subseteq \sigma(\mathcal{I}_i) \subseteq \mathcal{J}_i.$$

We can extend σ to an automorphism of $\mathcal{U}$. Let

$$A' = (\sigma^{-1}(a_0), \dots, \sigma^{-1}(a_m)).$$

Now any m-sized subset of A' inserts into $\mathcal{I}_0 + \cdots + \mathcal{I}_{m+1}$, but A' does not. ∎

Return

Proof of Lemma:

We will only handle the case where $\mathcal{I}$ is dense.

Assume no such A' exists.

By compactness, there are $\phi \in \mathrm{tp}_{\mathcal{I}}(a_0, \ldots, a_m)$ and $\psi_\sigma \in \mathrm{limtp}_D(\mathfrak{c}^-_{\sigma(0)}, \ldots, \mathfrak{c}^-_{\sigma(m-1)})$ for each $\sigma : m \to m+1$ increasing such that

$$\phi(x_0, \ldots, x_m) \vdash \bigvee_\sigma \neg\psi_\sigma(x_{\sigma(0)}, \ldots, x_{\sigma(m-1)}). \tag{$*$}$$

Let $B \subseteq \mathcal{I}$ be the parameters of ϕ.

For each σ as above, we construct an indiscernible sequence $\mathcal{J}_\sigma$ by induction...

Stage 0

For all $j < m+1$, choose $\mathcal{I}_j^0$ to be a proper end segment of $\mathcal{I}_j$ excluding B such that each ψ_σ is satisfied by every element of $\mathcal{I}^0_{\sigma(0)} \times \cdots \times \mathcal{I}^0_{\sigma(m-1)}$.

Let $\mathcal{I}^0 = \mathcal{I}$, and let $\mathcal{J}^0_\sigma = \emptyset$ for each σ.

Stage $2i + 1$

Let $\mathcal{I}'$ be a finite subset of $\mathcal{I}^{2i}$ containing B.

There is an increasing map

$$\mathcal{I}' \longrightarrow \mathcal{I} \setminus \bigcup_j \mathcal{I}_j^{2i}$$

fixing B such that for each $j < m + 1$, elements to the left of $\mathcal{I}_j^{2i}$ remain to the left and all other elements map to the right of $\mathcal{I}_j^{2i}$.

This map extends to an automorphism fixing B, so by compactness, there is $A' = (a'_0, \ldots, a'_m)$ realizing ϕ such that if we assign each a'_j to the cut of $\mathcal{I}^{2i}$ immediately to the left of $\mathcal{I}_j^{2i}$, then any proper subsequence of A' inserts into $\mathcal{I}^{2i} \supseteq \mathcal{I}$.

Stage $2i+1$ (continued)

Recall

$$\phi(x_0, \ldots, x_m) \vdash \bigvee_{\sigma} \neg\psi_\sigma(x_{\sigma(0)}, \ldots, x_{\sigma(m-1)}). \qquad (*)$$

We can choose $\sigma_i : m \to m+1$ increasing so that

$$a'_{\sigma_i(0)} \cdots a'_{\sigma_i(m-1)} \not\models \psi_{\sigma_i}.$$

Let

$$\mathcal{I}^{2i+1} = \mathcal{I}^{2i} \cup \left\{ a'_{\sigma_i(j)} \ : \ j < m \right\}$$

where each $a'_{\sigma_i(j)}$ is inserted immediately to the left of $\mathcal{I}^{2i}_{\sigma_i(j)}$. Let

$$\mathcal{J}^{2i+1}_{\sigma_i} = \mathcal{J}^{2i}_{\sigma_i} + \left(a'_{\sigma_i(0)}, \ldots, a'_{\sigma_i(m-1)} \right).$$

For each $j < m+1$, let $\mathcal{I}^{2i+1}_j = \mathcal{I}^{2i}_j$.

Stage $2i+2$

For each $j < m+1$, choose $b_j \in \mathcal{I}_j^{2i+1}$ and an end segment $\mathcal{I}_j^{2i+2}$ of $\mathcal{I}_j^{2i+1}$ excluding b_j.

Let $\mathcal{I}^{2i+2} = \mathcal{I}^{2i+1}$, and for each σ, let

$$\mathcal{J}_\sigma^{2i+2} = \mathcal{J}_\sigma^{2i+1} + \left(b_{\sigma(0)}, \ldots, b_{\sigma(m-1)}\right).$$

Proof of Lemma (continued)

For each σ, let $\mathcal{J}_\sigma = \bigcup_{i<\omega} \mathcal{J}^i_\sigma$.

Choose a σ which appears infinitely many times in $(\sigma_i : i < \omega)$.

It follows that ψ_σ alternates infinitely many times on $\mathcal{J}_\sigma$, contradicting NIP.

$\blacksquare$

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